Tutorial Note Mar. 26

1. Integrate the Taylor series expansion

$$\frac{1}{w} = \sum_{n=0}^{\infty} (-1)^n (w-1)^n, \qquad |w-1|<1$$

along a contour to obtain the following representation

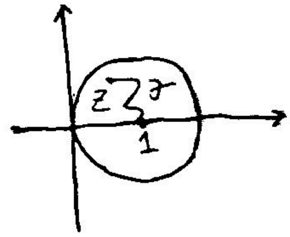
$$\operatorname{Log} z = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (z-1)^n, \quad |z-1|<1.$$

<u>Sol</u>. Recall the integration of power series:

" $\displaystyle\int_\gamma f(z)\,dz = \sum_{n=0}^{\infty} \int_\gamma a_n (z-z_0)^n\,dz$

where $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, and $\gamma \subseteq$ disc of convergence"

We take $\gamma \subseteq \{w, |w-1|<1\}$, connecting $w=1$ & $w=z$ for any z s.t. $|z-1|<1$. Then we apply the Fundamental Thm of complex line integrals



" If $F(z)$ is holomorphic on domain D and γ is a curve in D from z_1 to z_2, then $\int_\gamma F'(z)\,dz = F(z_2) - F(z_1)$ "

and the relation $(\operatorname{Log} z)' = \frac{1}{z}$, $((z-1)^{n+1})' = n(z-1)^n$ to yield

$$\operatorname{Log}(z) - \operatorname{Log}(1) = \int_\gamma \frac{1}{w}\,dw = \sum_{n=0}^{\infty} \int_\gamma (-1)^n (w-1)^n\,dw$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (w-1)^{n+1} \Big|_{w=1}^{w=z} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (z-1)^n$$

$\square$

2. Assume that $S(z) = \sum_{n=-\infty}^{\infty} C_n (z-z_0)^n$ is convergent in the annulus $A(z_0; r, R) = \{z,\ r < |z-z_0| < R\}$. Let contour $C \subseteq A(z_0; r, R)$, and $g(z)$ is a continuous function on C. Prove that

$$\int_C g(z) S(z)\, dz = \sum_{n=-\infty}^{\infty} C_n \int_C g(z)(z-z_0)^n dz.$$

Pf. Step 1. $\sum_{n=-\infty}^{\infty} C_n (z-z_0)^n$ converges absolutely on C

This follows directly from an analysis on the radius of convergence

Step 2. $\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ s.t. $\forall N > N_\varepsilon$,

$$\left| S(z) - \sum_{n=-N}^{N} C_n (z-z_0)^n \right| < \varepsilon \quad \text{for } \forall z \in C$$

Therefore,

$$\left| \int_C g(z) S(z) dz - \int \sum_{n=-N}^{N} C_n (z-z_0)^n g(z)\, dz \right|$$

$$= \left| \int_C g(z) \left(S(z) - \sum_{n=-N}^{N} C_n (z-z_0)^n \right) dz \right| \qquad (*)$$

$$\leq \max_{z \in C} |g(z)| \left| S(z) - \sum_{n=-N}^{N} C_n (z-z_0)^n \right| \cdot \text{Length}(C)$$

$$\leq \varepsilon \,\text{length}(C) \max_{z \in C} |g(z)|.$$

It follows from $(*)$ that the convergence hold. $\square$

3. Determine the order of the pole and find the corresponding residue.

(i) $f(z) = \dfrac{z^{1/4}}{z+1}$ $(|z|>0,\ 0<\arg z<2\pi)$ at $z=-1$

(ii) $f(z) = \dfrac{\operatorname{Log} z}{(z^2+1)^2}$, at $z = i$ & $-i$

<u>Sol</u>. Recall that z_0 is a pole of order m of f if

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}, \text{ where } \phi(z) \text{ is analytic \& nonzero at } z_0.$$

Moreover, $\displaystyle \operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$.

(i) -1 is a pole of order 1, and $\phi(z) = z^{\frac14}$, $(|z|>0,\ 0<\arg z<2\pi)$

$$\operatorname*{Res}_{z=-1} \frac{z^{\frac14}}{z+1} = \phi(-1) = \exp\left(\tfrac14\left(\ln|-1| + i\arg(-1)\right)\right)$$

$$= \exp\left(\tfrac14\left(0+\pi i\right)\right) = \exp\left(\tfrac{\pi}{4} i\right) = \frac{1+i}{\sqrt2}.$$

(ii) i is a pole of order 2, and $\phi(z) = \dfrac{\operatorname{Log} z}{(z+i)^2}$

$$\operatorname*{Res}_{z=i} \frac{\operatorname{Log} z}{(z^2+1)^2} = \frac{\phi'(i)}{(2-1)!} = \frac{1}{z(z+i)^2} - \frac{2\operatorname{Log} z}{(z+i)^3}\Bigg|_{z=i}$$

$$= \frac{1}{i(2i)^2} - \frac{2\operatorname{Log} i}{(2i)^3} = \frac{i}{4} - \frac{i}{4}\operatorname{Log} i = \frac{i}{4} - \frac{i}{4}\,\frac{\pi}{2} i = \frac{i}{4} + \frac{\pi}{8}$$

$-i$ is a pole of order 2, and $\phi(z) = \dfrac{\operatorname{Log} z}{(z-i)^2}$

$$\operatorname*{Res}_{z=-i} \frac{\operatorname{Log} z}{(z^2+1)^2} = \phi'(-i) = \frac{1}{z(z-i)^2} - \frac{2\operatorname{Log} z}{(z-i)^3}\Bigg|_{z=-i}$$

$$= \frac{1}{(-i)(-2i)^2} - \frac{2\operatorname{Log}(-i)}{(-2i)^3} = -\frac{i}{4} + \frac{i}{4}\left(-\frac{\pi}{2} i\right) = \frac{\pi}{8} - \frac{i}{4}$$

$\square$

4. Let p & q be analytic at z_0, $p(z_0) \neq 0$, $q(z_0) = 0$. Show that if $\frac{p(z)}{q(z)}$ has a pole of order m at z_0, then z_0 is a zero of order m of q.

<u>Pf.</u> Recall z_0 is a zero of order m of q if

$$q(z) = (z-z_0)^m \left(a_m + a_{m+1}(z-z_0) + \dots \right) \qquad (**)$$

where $a_m \neq 0$.

It follows from the definition of the order of a pole that

$$\frac{p(z)}{q(z)} = \frac{\phi(z)}{(z-z_0)^m},$$ where $\phi(z)$ is analytic & nonzero at z_0.

Therefore $q(z) = (z-z_0)^m \frac{\phi(z)}{p(z)}$.

Here $\frac{\phi(z)}{p(z)}$ is analytic at z_0 and $\frac{\phi(z_0)}{p(z_0)} \neq 0$.

Therefore, one can obtain the following series expansion at z_0:

$$\frac{\phi(z)}{p(z)} = \sum_{n=0}^{\infty} b_n (z-z_0)^n, \qquad b_0 \neq 0.$$

Thus, $q(z) = (z-z_0)^m \sum_{n=0}^{\infty} b_n (z-z_0)^n$, it follows from (**) that q has an m order zero at z_0. $\square$